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Obtaining a Function of Bounded Coarse Variation by a Change of Variable

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Recently two new integrals were defined, on $[0, \infty)$ and $(0, 1]$, respectively (see [1, 2].). They are the simple integral and the dominated integral. Each integral is defined by "one limit," which is somewhat unusual for an "improper" integral. When either of these two integrals is defined, it agrees with the improper Riemann integral of the function on $[0, \infty)$ or $(0, 1]$, respectively, which must then exist. Since the class of improperly Riemann integrable functions on $[0, \infty)$ or $(0, 1]$ is larger than the class of simply or dominantly integrable functions, respectively, it is of interest to see how and when one can change an improperly Riemann integrable function $F(x)$ into either a simply or dominantly integrable function $F(f(t))\, f'(t)$ via a change of variable $x = f(t)$. We treat this question below.

DEFINITIONS 1 (From [1]). If ε is a positive number, a set of real numbers is called "ε-separated" when every two numbers in the set differ by ε or more.

If f is a complex function on $[0, \infty)$ and $\{x_0, x_1, x_2,...\}$ is a finite or infinite, strictly increasing, sequence of nonnegative numbers, the (finite or infinite) quantity

$$\sum_j |f(x_j) - f(x_{j-1})|$$

is called "the variation of f on the sequence $\{x_0, x_1,...\}$." If S is a (nonempty) set of nonnegative real numbers with no finite limit point, and S^* the sequence consisting of the elements of S in their natural order, then the "variation of f on S" is just the variation of f on S^*.

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For a complex function f on $[0, \infty)$ and $\varepsilon > 0$, the "ε-variation of f" (denoted $V_\varepsilon(f)$) is the supremum of the variations of f on all ε-separated sets of nonnegative real numbers. (Clearly to define $V_\varepsilon(f)$, it suffices to consider only finite ε-separated sets.)

A complex function f on $[0, \infty)$ is said to be of "bounded coarse variation" (BCV) if it has a finite ε-variation for every $\varepsilon > 0$.

THEOREM 1. *Let F be a complex function on* $[0, \infty)$, *Riemann integrable on every* $[0, M]$, $0 < M < \infty$. *Then there exists a continuously differentiable real function f from* $[0, \infty)$ *onto* $[0, \infty)$, *with* $f' > 0$ *on* $[0, \infty)$, *such that* $F(f(t))\, f'(t)$ *is of bounded coarse variation (by* $f'(0)$ *we mean a right-hand derivative).*

Remark. The proof which we shall give is constructive. For many functions F, at least one such f has a relatively simple form (see Theorem 4).

A partition Π of an interval $[0, b]$, $0 < b < \infty$, is a sequence $0 = s_0 < s_1 < \cdots < s_m = b$ $(m \geqslant 1)$. The mesh of Π is defined to be $\max_{0 \leqslant j \leqslant m-1} (s_{j+1} - s_j)$ and is denoted by $|\Pi|$.

DEFINITION 2 (From [1]). A complex function f on $[0, \infty)$ is called "simply integrable" (over $[0, \infty)$) if there is a number I such that: For every $\varepsilon > 0$ there are positive numbers $B = B(\varepsilon)$ and $\Delta = \Delta(\varepsilon)$ such that if $b > B$, Π is a partition of $[0, b]$ with $|\Pi| < \Delta$ and Σ is a Riemann sum for f, based on Π, then $|\Sigma - I| < \varepsilon$.

In other words, f is simply integrable if the Riemann sums associated with partitions Π of $[0, b]$ approach a unique (finite) limit as long as $b \to \infty$ and $|\Pi| \to 0$ simultaneously.

THEOREM 2. *Let F, f be as in Theorem* 1 *and suppose also that the improper Riemann integral* $\int_0^\infty F(t)\, dt$ *converges. Then* $G(t) \equiv F(f(t))\, f'(t)$ *is simply integrable.*

DEFINITION 3 (From [2]). Let H be a complex function on $(0, 1]$. A dominated integral of H is a complex number $I(H)$ having the following property:

For each $\varepsilon > 0$ there exist δ and χ, $0 < \delta < 1$, $0 < \chi < 1$, such that

$$\left| I(H) - \sum_{j=1}^{n} H(\tau_j)(t_j - t_{j-1}) \right| < \varepsilon \tag{1}$$

whenever $0 < t_0 < t_1 < \cdots < t_n = 1$, $t_0 < \chi$, $t_{j-1} \leqslant \tau_j \leqslant t_j$, and $t_{j-1} t_j^{-1} > 1 - \delta$, $j = 1, 2, \ldots, n$.

The existence of a dominated integral for a complex function H on $(0, 1]$ (H being "dominantly integrable") implies that H is properly Riemann integrable on $[\varepsilon, 1]$ for each ε in $(0, 1)$ and that $\int_0^\infty H(t)\,dt = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 H(t)\,dt$ exists, is finite, and equals $I(H)$. As with the simple integral, the dominated integral is defined by a "one limit" procedure and its existence implies, see [3], that many other procedures can be validly used to numerically approximate $\int_0^\infty H(t)\,dt$.

THEOREM 3. *Suppose that H is a complex function on $(0, 1]$ which is Riemann integrable on $(\varepsilon, 1]$ for each ε in $(0, 1)$, and that $\int_0^\infty |H(t)|\,dt = \lim_{\varepsilon \to 0+} \int_\varepsilon^1 |H(t)|\,dt < \infty$. Then there is a continuously differentiable function h from $(0, 1]$ onto $(0, 1]$, with $h' > 0$ on $(0, 1]$, such that $H(h(t))\,h'(t)$ is dominantly integrable.*[1]

THEOREM 4. *Let F be a complex function on $[0, \infty)$ and let $K \geqslant 1$ and $\alpha > 0$ be constants such that, for $n = 0, 1, 2,...,$*

(i) *the total variation of F on $[n, n+2] \leqslant K(n+1)^\alpha$ and*

(ii) $\sup_{0 \leqslant t \leqslant n+1} |F(t)| \leqslant K(n+1)^\alpha$.

Then $(1 + K^{-1}t)^{1/(3+\alpha)} - 1$ can serve as f of Theorem 1.

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DEFINITION 4. If f is a bounded complex function on some $[a, b]$ we denote by $w(f, a, b)$ the oscillation of f on $[a, b]$, i.e., the supremum of the set of all $|f(t_1) - f(t_2)|$ with $a \leqslant t_1 \leqslant t_2 \leqslant b$. Given a complex function $f(t)$ defined and bounded on each closed subinterval of $(0, 1]$, and given a sequence $0 < t_0 < t_1 < \cdots < t_n = 1$, let $\mathrm{OS}(f; t_0,..., t_n)$ denote the oscillation sum

$$\sum_{j=1}^{n} w(f, t_{j-1}, t_j)(t_j - t_{j-1}).$$

Proofs of Theorems 1 *and* 4. Since F is Riemann integrable on each $[0, M]$, $0 < M < \infty$, F is bounded on each such interval so that $\hat{F}(x) = \sup_{0 \leqslant t \leqslant x} |F(t)| < \infty$, $0 \leqslant x < \infty$.

Let $(\delta(n))_{n=0}^\infty$ be a sequence of positive numbers so that any oscillation sum for F on $[n, n+2]$ with mesh $\leqslant \delta(n)$ is $\leqslant (n+1)^{-2}$, $n = 0, 1, 2,....$

We next observe that there exists a function φ with the following properties: φ is a positive, strictly decreasing, continuous function on $[0, \infty)$;

[1] $h'(1)$ is a left-hand derivative.

$\varphi(0) \leqslant 1$; $\lim_{x\to\infty} \varphi(x) = 0$; for $n = 0, 1, \ldots$, $\varphi(n) \leqslant \delta(n)$; and on each $(n, n+1)$, φ is differentiable and $|\varphi'| \leqslant ((n+1)^2 \hat{F}(n+1))^{-1}$ (here $0^{-1} = +\infty$).

For example, for $n = 1, 2, \ldots$, $\varphi(x)$ can be taken to the linear on $[n-1, n)$ with $\varphi(0) = \min\{1, \delta(0), (\hat{F}(1))^{-1}\}$ and $\varphi(n) = \min\{\delta(n), \frac{1}{2}\varphi(n-1), ((n+1)^2 \hat{F}(n+1))^{-1}\}$. Notice that if $n < x < n+1$, $|\varphi'(x)| = \varphi(n) - \varphi(n+1) < \varphi(n) \leqslant ((n+1)^2 \hat{F}(n+1))^{-1}$. Under the hypotheses of Theorem 4, we can set $\delta(n) \equiv (2K(n+1)^{\alpha+2})^{-1}$ and $\varphi(x) \equiv ((\alpha+3) K(x+1)^{\alpha+2})^{-1}$. We notice that $|\varphi'(x)| < K^{-1}(x+1)^{-(\alpha+3)} \leqslant (n+1)^{-2} K^{-1}(n+1)^{-\alpha}$, if $n \leqslant x \leqslant n+1$, $n = 0, 1, 2, \ldots$.

Since $\lim_{x\to\infty} \varphi(x) = 0$, $\lim_{x\to\infty} \int_0^\infty (\varphi(t))^{-1}\, dt = \infty$. On $[0, \infty)$, let $g(x) = \int_0^x (\varphi(t))^{-1}\, dt$ and let $f(x)$ be defined by $g(f(x)) = x$. Clearly

$$f'(g(x)) = (g'(x))^{-1} = \varphi(x) \tag{2}$$

if $x \geqslant 0$ ($f'(0)$ and $g'(0)$ are right-hand derivatives).

Notice that, under the hypotheses of Theorem 4, $g(x) \equiv K(x+1)^{\alpha+3} - K$ and $f(t) \equiv (1 + K^{-1}t)^{1/(3+\alpha)} - 1$.

Consider sums of the form

$$\sum_{j=2}^{N} |F(f(x_j))\, f'(x_j) - F(f(x_{j-1}))\, f'(x_{j-1})|, \tag{3}$$

where $0 \leqslant x_1 < x_2 < \cdots < x_N < \infty$ and $x_j - x_{j-1} \geqslant \frac{1}{2}$, $j = 2, 3, \ldots, N \geqslant 2$. If we can uniformly bound all sums (3), then $F(f(x))\, f'(x)$ will have been shown to be of bounded ε-variation for $\varepsilon = \frac{1}{2}$. By Theorem 1 of [4] this will suffice to show that $F(f(x))\, f'(x)$ is of bounded ε-variation for each $\varepsilon > 0$. Clearly we may also require above that $x_j - x_{j-1} \leqslant 1$, for $j = 2, 3, \ldots, N$. Now

$$\begin{aligned} &F(f(x_j))\, f'(x_j) - F(f(x_{j-1}))\, f'(x_{j-1}) \\ &\quad = [F(f(x_j)) - F(f(x_{j-1}))]\, f'(x_j) + F(f(x_{j-1}))(f'(x_j) - f'(x_{j-1})). \end{aligned}$$

We shall proceed to bound both

$$\sum_{j=2}^{N} |F(f(x_{j-1}))|\, |f'(x_j) - f'(x_{j-1})| \tag{4}$$

and

$$\sum_{j=2}^{N} |F(f(x_j)) - F(f(x_{j-1}))|\, f'(x_j). \tag{5}$$

Setting $y_j = f(x_j)$, $1 \leqslant j \leqslant N$, we have

$$\sum_{j=2}^{N} |F(f(x_{j-1}))|\,|f'(x_j)-f'(x_{j-1})|$$
$$=\sum_{j=2}^{N} |F(y_{j-1})|\,|f'(g(y_j))-f'(g(y_{j-1}))|$$
$$=\sum_{j=2}^{N} |F(y_{j-1})|\,|\varphi(y_j)-\varphi(y_{j-1})|$$

(recall (2)). For some ξ_j, $y_{j-1}<\xi_j<y_j$, $2\leqslant j\leqslant N$, by the mean value theorem,

$$y_j-y_{j-1}=(x_j-x_{j-1})\,f'(g(\xi_j))=(x_j-x_{j-1})\,\varphi(\xi_j)$$
$$\leqslant\varphi(\xi_j)\leqslant\varphi(0)\leqslant 1.$$

A consequence of the mean value theorem is that, if u is a real function, on some $[a,b]$ $(-\infty<a<b<\infty)$ and differentiable on (a,b), then

$$|f(b)-f(a)|\leqslant(b-a)\sup_{t\in(a,b)}|f'(t)|.$$

If u is continuous on $[a,b]$ and only piecewise differentiable on (a,b) and if we interpret $\sup_{t\in(a,b)}|f'(t)|$ as the sup over those t for which $f'(t)$ exists, then the above inequality continues to hold.

Therefore

$$\sum_{j=2}^{N} |F(y_{j-1})|\,|\varphi(y_j)-\varphi(y_{j-1})|$$
$$=\sum_{k=1}^{\infty}\sum_{y_{j-1}\in[k-1,k)} |F(y_{j-1}|\,|\varphi(y_j)-\varphi(y_{j-1})|$$
$$\leqslant\sum_{k=1}^{\infty}\left(\sum_{y_{j-1}\in[k-1,k)}|\varphi(y_j)-\varphi(y_{j-1})|\right)\hat{F}(k)$$
$$\leqslant\sum_{k=1}^{\infty}2(\sup_{k-1<t<k+1}|\varphi'(t)|)\,\hat{F}(k)$$
$$\leqslant 2\sum_{k=1}^{\infty}k^{-2}<\infty.$$

Now we consider (5). Label the points $y_j=f(x_j)$ lying in $[m-1,m+1)$, for $j=1,2,\ldots,N$, $m=1,2,\ldots$, if any, as $\tau_{k,m}$, where $\tau_{1,m}<\tau_{2,m}<\cdots<\tau_{K(m),m}$, $k(m)\geqslant 1$. Recalling (2), we can bound (5) by

$$\sum_{m=1}^{\infty}\sum_{k=1}^{K(m)-1}|F(\tau_{k+1,m})-F(\tau_{k,m})|\,\varphi(\tau_{k+1,m})$$

(an empty sum is 0). Let $m \geqslant 1$, $K(m) > 1$. If $1 \leqslant k < k+1 \leqslant K(m)$, then for some $\xi_{k,m}$, $\tau_{k,m} < \xi_{k,m} < \tau_{k+1,m}$, one has $(\tau_{k+1,m} - \tau_{k,m})(g(\tau_{k+1,m}) - g(\tau_{k,m}))^{-1} = f'(g(\xi_{k,m})) = \varphi(\xi_{k,m}) > \varphi(\tau_{k+1,m})$. Since each $x_j - x_{j-1} \geqslant \frac{1}{2}$, we see that

$$\sum_{k=1}^{K(m)-1} |F(\tau_{k+1,m}) - F(\tau_{k,m})| \, \varphi(\tau_{k+1,m})$$
$$\leqslant 2 \sum_{k=1}^{K(m)-1} |F(\tau_{k+1,m}) - F(\tau_{k,m})| \, (\tau_{k+1,m} - \tau_{k,m}).$$

If $1 \leqslant k < k+1 \leqslant K(m)$, let $O_{k,m}$ denote the oscillation of F on $[\tau_{k,m}, \tau_{k+1,m}]$. Then

$$\sum_{k=1}^{K(m)-1} |F(\tau_{k+1,m}) - F(\tau_{k,m})| \, (\tau_{k+1,m} - \tau_{k,m})$$
$$\leqslant \sum_{k=1}^{K(m)-1} O_{k,m}(\tau_{k+1,m} - \tau_{k,m}).$$

In the last sum, each $\tau_{k+1,m} - \tau_{k,m} \leqslant \max_{2 \leqslant j \leqslant N}(y_j - y_{j-1}) \leqslant 1$. Thus for $m = 1, ...,$ $\sum_{k=1}^{K(m)-1} O_{k,m}(\tau_{k+1,m} - \tau_{k,m})$ can be regarded as a subsum of an oscillation sum for F on $[m-1, m+1]$ with mesh $\leqslant 1$. Since $\max_{2 \leqslant j \leqslant N}(x_j - x_{j-1}) \leqslant 1$, we have, for $k = 1, 2, ..., K(m)-1$, $\tau_{k+1,m} - \tau_{k,m} = (g(\tau_{k+1,m}) - g(\tau_{k,m})) \, \varphi(\xi_{k,m}) \leqslant \varphi(\xi_{k,m}) \leqslant \delta(m-1)$. It follows by the choice of $\delta(m-1)$, that $\sum_{k=1}^{K(m)-1} O_{k,m}(\tau_{k+1,m} - \tau_{k,m}) < m^{-2}$. This proves Theorems 1 and 4.

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Proof of Theorem 2. By Theorem 3 of [1], G is simply integrable since it is improperly Riemann integrable over $[0, \infty)$ and of bounded coarse variation.

Proof of Theorem 3. $F(t) \equiv H((t+1)^{-1})(t+1)^{-2}$, $0 \leqslant t < \infty$, satisfies the hypothesis of Theorem 1. Thus there exists f as in Theorem 1 such that $G(t) = F(f(t)) f'(t)$, and a fortiori $|G(t)|$ is of bounded coarse variation, as clearly $\int_0^\infty |G(t)| \, dt < \infty$. By Theorem 3 of [1], $|G|$ is simply integrable. Now G is Riemann integrable on each $[0, M]$, $0 < M < \infty$, and thus it is "absolutely simply integrable" ([2, Definition 3]). By Theorem 5 of [2], $G(-\log t) \, t^{-1}$ is dominantly integrable. Set $h(t) \equiv (1 + f(-\log t))^{-1}$, $0 < t \leqslant 1$. The function satisfies the conclusion of Theorem 3.

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